CHAPTER

# 2 Limits and Continuity



An Economic Injury Level (EIL) is a measurement of the fewest number of insect pests that will cause economic damage to a crop or forest. It has been estimated that monitoring pest populations and establishing EILs can reduce pesticide use by 30%–50%.

Accurate population estimates are crucial for determining EILs. A population density of one insect pest can be approximated by

$$D(t) = \frac{t^2}{90} + \frac{t}{3}$$

pests per plant, where $t$ is the number of days since initial infestation. What is the rate of change of this population density when the population density is equal to the EIL of 20 pests per plant? Section 2.4 can help answer this question.





## CHAPTER 2 Overview

The concept of limit is one of the ideas that distinguish calculus from algebra and trigonometry.

In this chapter, we show how to define and calculate limits of function values. The calculation rules are straightforward, and most of the limits we need can be found by substitution, graphical investigation, numerical approximation, algebra, or some combination of these.

One of the uses of limits is to test functions for continuity. Continuous functions arise frequently in scientific work because they model such an enormous range of natural behavior. They also have special mathematical properties, not otherwise guaranteed.

# 2.1 Rates of Change and Limits

**What you will learn about . . .**

- Average and Instantaneous Speed
- Definition of Limit
- Properties of Limits
- One-Sided and Two-Sided Limits
- Sandwich Theorem

**and why . . .**

Limits can be used to describe continuity, the derivative, and the integral: the ideas giving the foundation of calculus.

## Average and Instantaneous Speed

Average speed of a moving body during an interval of time is found by dividing the distance covered by the elapsed time. More precisely, if $y = f(t)$ is a distance or position function of a moving body at time $t$, then the **average rate of change** (or **average speed**) is the ratio

$$\frac{f(t+h) - f(t)}{h}$$

where the elapsed time is the interval of time from $t$ to $t + h$, or simply $h$. Often we call the numerator $\Delta y$, the change in $y = f(t)$, and the denominator $\Delta t$, the change in elapsed time $t$, where $\Delta y$ is read "delta $y$" and $\Delta t$ is read "delta $t$." Thus we often write the ratio as

$$\frac{\Delta y}{\Delta t} = \frac{f(t+h) - f(t)}{h}.$$

**Free Fall**

Near the surface of the earth, all bodies fall with the same constant acceleration. The distance a body falls after it is released from rest is a constant multiple of the square of the time fallen. At least, that is what happens when a body falls in a vacuum, where there is no air to slow it down. The square-of-time rule also holds for dense, heavy objects like rocks, ball bearings, and steel tools during the first few seconds of fall through air, before the velocity builds up to where air resistance begins to matter. When air resistance is absent or insignificant and the only force acting on a falling body is the force of gravity, we call the way the body falls *free fall.*

### EXAMPLE 1 Finding an Average Speed

A rock breaks loose from the top of a tall cliff. What is its average speed during the first 2 seconds of fall?

**SOLUTION**

Experiments show that a dense solid object dropped from rest to fall freely near the surface of the earth will fall

$$y = 16t^2$$

feet in the first $t$ seconds. The average speed of the rock over any given time interval is the distance traveled, $\Delta y$, divided by the length of the interval $\Delta t$. For the first 2 seconds of fall, from $t = 0$ to $t = 2$, we have

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\,\frac{\text{ft}}{\text{sec}}.$$

*Now Try Exercise 1.*

A moving body's **instantaneous speed** is the speed of the moving object at a *given instant of time*. The major issue is *how* to compute this speed, since the "elapsed time" seems to be zero, as we explain in Example 2. We show after this example that we need the mathematical concept of a *limit* to understand and compute instantaneous speed.

**TABLE 2.1 Average Speeds over Short Time Intervals Starting at $t = 2$**

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}$$

| Length of Time Interval, $h$ (sec) | Average Speed for Interval $\Delta y/\Delta t$ (ft/sec) |
|---|---|
| 1 | 80 |
| 0.1 | 65.6 |
| 0.01 | 64.16 |
| 0.001 | 64.016 |
| 0.0001 | 64.0016 |
| 0.00001 | 64.00016 |

**EXAMPLE 2 Finding an Instantaneous Speed**

Find the speed of the rock in Example 1 at the instant $t = 2$.

**SOLUTION**

**Solve Numerically** We can calculate the average speed of the rock over the interval from time $t = 2$ to any slightly later time $t = 2 + h$ as

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}. \qquad (1)$$

We cannot use this formula to calculate the speed at the exact instant $t = 2$ because that would require taking $h = 0$, and $0/0$ is undefined. However, we can get a good idea of what is happening at $t = 2$ by evaluating the formula at values of $h$ *close* to 0. When we do, we see a clear pattern (Table 2.1). As $h$ approaches 0, the average speed approaches the limiting value 64 ft/sec.

**Confirm Algebraically** If we expand the numerator of Equation 1 and simplify, we find that

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h} = \frac{16(4+4h+h^2) - 64}{h}$$
$$= \frac{64h + 16h^2}{h} = 64 + 16h.$$

For values of $h$ different from 0, the expressions on the right and left are equivalent and the average speed is $64 + 16h$ ft/sec. We can now see why the average speed has the limiting value $64 + 16(0) = 64$ ft/sec as $h$ approaches 0. ***Now Try Exercise 3.***

## Definition of Limit

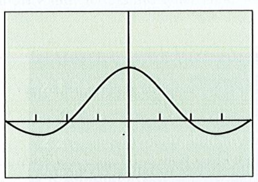
As in the preceding example, most limits of interest in the real world can be viewed as numerical limits of values of functions. And this is where a graphing utility and calculus come in. A calculator can suggest the limits, and calculus can give the mathematics for confirming the limits analytically.

Limits give us a language for describing how the outputs of a function behave as the inputs approach some particular value. In Example 2, the average speed was not defined at $h = 0$ but approached the limit 64 as $h$ approached 0. We were able to see this numerically and to confirm it algebraically by eliminating $h$ from the denominator. But we cannot always do that. For instance, we can see both graphically and numerically (Figure 2.1) that the values of $f(x) = (\sin x)/x$ approach 1 as $x$ approaches 0.

We cannot eliminate the $x$ from the denominator of $(\sin x)/x$ to confirm the observation algebraically. We need to use a theorem about limits to make that confirmation, as you will see in Exercise 77.

**Formal Definition of Limit**

You may want to look over the examples in Appendix A3, pp. 577–584, which provide illustrations of the formal limit definition of a function **$f$ that has limit $L$ as $x$ approaches $c$.**



$[-2\pi, 2\pi]$ by $[-1, 2]$

(a)

| X | Y1 |
|---|---|
| -.3 | .98507 |
| -.2 | .99335 |
| -.1 | .99833 |
| 0 | ERROR |
| .1 | .99833 |
| .2 | .99335 |
| .3 | .98507 |

Y1 = sin(X)/X

(b)

**Figure 2.1** (a) A graph and (b) table of values for $f(x) = (\sin x)/x$ that suggest the limit of $f$ as $x$ approaches 0 is 1.

**DEFINITION Limit**

Assume $f$ is defined in a neighborhood of $c$ and let $c$ and $L$ be real numbers. The function **$f$ has limit $L$ as $x$ approaches $c$** if, given any positive number $\varepsilon$, there is a positive number $\delta$ such that for all $x$,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

We write

$$\lim_{x\to c} f(x) = L.$$



The sentence $\lim_{x\to c} f(x) = L$ is read, "The limit of $f$ of $x$ as $x$ approaches $c$ equals $L$." The notation means that the values $f(x)$ of the function $f$ approach or equal $L$ as the values of $x$ approach *(but do not equal)* $c$. Appendix A3 provides practice applying the definition of limit.

We saw in Example 2 that $\lim_{h\to 0}(64 + 16h) = 64$.

As suggested in Figure 2.1,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Figure 2.2 illustrates the fact that the existence of a limit as $x \to c$ never depends on how the function may or may not be defined at $c$. The function $f$ has limit 2 as $x \to 1$ even though $f$ is not defined at 1. The function $g$ has limit 2 as $x \to 1$ even though $g(1) \neq 2$. The function $h$ is the only one whose limit as $x \to 1$ equals its value at $x = 1$.

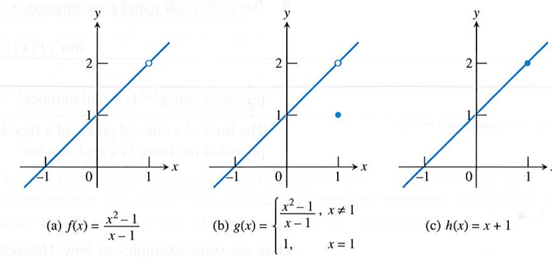



**Figure 2.2** $\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x) = \lim_{x\to 1} h(x) = 2$.

## Properties of Limits

By applying six basic facts about limits, we can calculate many unfamiliar limits from limits we already know. For instance, from knowing that

$$\lim_{x\to c} (k) = k \qquad \text{Limit of the function with constant value } k$$

and

$$\lim_{x\to c} (x) = c, \qquad \text{Limit of the identity function at } x = c$$

we can calculate the limits of all polynomial and rational functions. The facts are listed in Theorem 1.

**THEOREM 1 Properties of Limits**

If $L$, $M$, $c$, and $k$ are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \text{ then}$$

**1.** *Sum Rule:* $\lim_{x\to c} (f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits.

**2.** *Difference Rule:* $\lim_{x\to c} (f(x) - g(x)) = L - M$

The limit of the difference of two functions is the difference of their limits.

*continued*

3. *Product Rule:* $\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M$

   The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x\to c}(k\cdot f(x)) = k\cdot L$

   The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule:* $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$

   The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule:* If $r$ and $s$ are integers, $s \neq 0$, then

$$\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$$

   provided that $L^{r/s}$ is a real number.

   The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

Here are some examples of how Theorem 1 can be used to find limits of polynomial and rational functions.

**Using Analytic Methods**

We remind the student that *unless otherwise stated* all examples and exercises are to be done using analytic algebraic methods *without* the use of graphing calculators or computer algebra systems.

### EXAMPLE 3 Using Properties of Limits

Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$, and the properties of limits to find the following limits.

(a) $\lim_{x\to c}(x^3 + 4x^2 - 3)$ (b) $\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5}$

**SOLUTION**

(a) $\lim_{x\to c}(x^3 + 4x^2 - 3) = \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3$ — Sum and Difference Rules

$= c^3 + 4c^2 - 3$ — Product and Constant Multiple Rules

(b) $\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x\to c}(x^4 + x^2 - 1)}{\lim_{x\to c}(x^2 + 5)}$ — Quotient Rule

$= \frac{\lim_{x\to c} x^4 + \lim_{x\to c} x^2 - \lim_{x\to c} 1}{\lim_{x\to c} x^2 + \lim_{x\to c} 5}$ — Sum and Difference Rules

$= \frac{c^4 + c^2 - 1}{c^2 + 5}$ — Product Rule

*Now Try Exercises 5 and 6.*

Example 3 shows the remarkable strength of Theorem 1. From the two simple observations that $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$, we can immediately work our way to limits of polynomial functions and most rational functions using substitution.



**THEOREM 2 Polynomial and Rational Functions**

1. If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is any polynomial function and $c$ is any real number, then

$$\lim_{x\to c} f(x) = f(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

2. If $f(x)$ and $g(x)$ are polynomials and $c$ is any real number, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}, \quad \text{provided that } g(c) \neq 0.$$

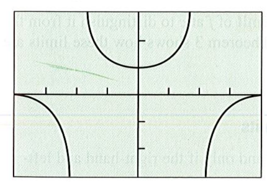
### EXAMPLE 4 Using Theorem 2

(a) $\lim_{x\to 3}[x^2(2 - x)] = (3)^2(2 - 3) = -9$

(b) $\lim_{x\to 2}\frac{x^2 + 2x + 4}{x + 2} = \frac{(2)^2 + 2(2) + 4}{2 + 2} = \frac{12}{4} = 3$

*Now Try Exercises 9 and 11.*

As with polynomials, limits of many familiar functions can be found by substitution at points where they are defined. This includes trigonometric functions, exponential and logarithmic functions, and composites of these functions. Feel free to use these properties.



[$-\pi$, $\pi$] by [−3, 3]

**Figure 2.3** The graph of

$f(x) = (\tan x)/x$

suggests that $f(x) \to 1$ as $x \to 0$. (Example 5)

### EXAMPLE 5 Using the Product Rule

Determine $\lim_{x\to 0}\frac{\tan x}{x}$.

**SOLUTION**

**Solve Analytically** Using the analytic result of Exercise 77, we have

$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right)$ — $\tan x = \frac{\sin x}{\cos x}$

$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}$ — Product Rule

$= 1\cdot\frac{1}{\cos 0} = 1\cdot\frac{1}{1} = 1.$

**Support Graphically** The graph of $f(x) = (\tan x)/x$ in Figure 2.3 suggests that the limit exists and is about 1.

*Now Try Exercise 33.*

Sometimes we can use a graph to discover that limits do not exist, as illustrated by Example 6.

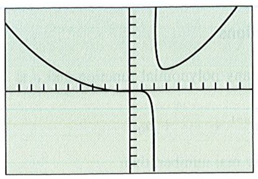
### EXAMPLE 6 Exploring a Nonexistent Limit

Use a graph to show that

$$\lim_{x\to 2}\frac{x^3 - 1}{x - 2}$$

does not exist.

*continued*



[−10, 10] by [−100, 100]

**Figure 2.4** The graph of $f(x) = (x^3 - 1)/(x - 2)$. (Example 6)

**SOLUTION**

Notice that the denominator is 0 when $x$ is replaced by 2, so we cannot use substitution to determine the limit. The graph in Figure 2.4 of $f(x) = (x^3 - 1)/(x - 2)$ strongly suggests that as $x \to 2$ from either side, the absolute values of the function values get very large. This, in turn, suggests that the limit does not exist.

***Now Try Exercise 35.***

## One-Sided and Two-Sided Limits

Sometimes the values of a function $f$ tend to different limits as $x$ approaches a number $c$ from opposite sides. When this happens, we call the limit of $f$ as $x$ approaches $c$ from the right the **right-hand limit** of $f$ at $c$ and the limit as $x$ approaches $c$ from the left the **left-hand limit** of $f$ at $c$. Here is the notation we use:

right-hand: $\lim_{x \to c^+} f(x)$ *The limit of f as x approaches c from the right.*

left-hand: $\lim_{x \to c^-} f(x)$ *The limit of f as x approaches c from the left.*

### EXAMPLE 7 Function Values Approach Two Numbers

The greatest integer function $f(x) = \text{int } x$ has different right-hand and left-hand limits at each integer, as we can see in Figure 2.5. For example,

$$\lim_{x \to 3^+} \text{int } x = 3 \quad \text{and} \quad \lim_{x \to 3^-} \text{int } x = 2.$$

The limit of int $x$ as $x$ approaches an integer $n$ from the right is $n$, while the limit as $x$ approaches $n$ from the left is $n - 1$.

***Now Try Exercises 37 and 38.***

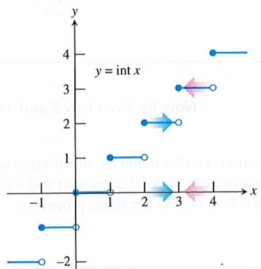



**Figure 2.5** At each integer, the greatest integer function $y = \text{int } x$ has different right-hand and left-hand limits. (Example 7)

We sometimes call $\lim_{x \to c} f(x)$ the **two-sided limit** of $f$ at $c$ to distinguish it from the *one-sided* right-hand and left-hand limits of $f$ at $c$. Theorem 3 shows how these limits are related.

**On the Far Side**

If $f$ is not defined to the left of $x = c$, then $f$ does not have a left-hand limit at $c$. Similarly, if $f$ is not defined to the right of $x = c$, then $f$ does not have a right-hand limit at $c$.

**THEOREM 3 One-Sided and Two-Sided Limits**

A function $f(x)$ has a limit as $x$ approaches $c$ if and only if the right-hand and left-hand limits at $c$ exist and are equal. In symbols,

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L.$$

Thus, the greatest integer function $f(x) = \text{int } x$ of Example 7 does not have a limit as $x \to 3$ even though each one-sided limit exists.

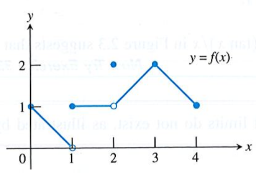



**Figure 2.6** The graph of the function

$$f(x) = \begin{cases} -x + 1, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \\ x - 1, & 2 < x \le 3 \\ -x + 5, & 3 < x \le 4. \end{cases}$$

(Example 8)

### EXAMPLE 8 Exploring Right- and Left-Hand Limits

All the following statements about the function $y = f(x)$ graphed in Figure 2.6 are true.

At $x = 0$: $\lim_{x \to 0^+} f(x) = 1$.

At $x = 1$: $\lim_{x \to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x \to 1^+} f(x) = 1$,

$f$ has no limit as $x \to 1$. (The right- and left-hand limits at 1 are not equal, so $\lim_{x \to 1} f(x)$ does not exist.)

*continued*



At $x = 2$: $\lim_{x \to 2^-} f(x) = 1$,

$\lim_{x \to 2^+} f(x) = 1$,

$\lim_{x \to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 2 = f(3) = \lim_{x \to 3} f(x)$.

At $x = 4$: $\lim_{x \to 4^-} f(x) = 1$.

At noninteger values of $c$ between 0 and 4, $f$ has a limit as $x \to c$.

***Now Try Exercise 43.***

## Sandwich Theorem

If we cannot find a limit directly, we may be able to find it indirectly with the Sandwich Theorem. The theorem refers to a function $f$ whose values are sandwiched between the values of two other functions, $g$ and $h$. If $g$ and $h$ have the same limit as $x \to c$, then $f$ has that limit too, as suggested by Figure 2.7.

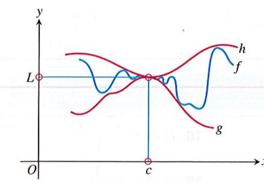


**Figure 2.7** Sandwiching $f$ between $g$ and $h$ forces the limiting value of $f$ to be between the limiting values of $g$ and $h$.

**THEOREM 4 The Sandwich Theorem**

If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about $c$, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$

### EXAMPLE 9 Using the Sandwich Theorem

Show that $\lim_{x \to 0} [x^2 \sin(1/x)] = 0$.

**SOLUTION**

We know that the values of the sine function lie between −1 and 1. So, it follows that

$$\left| x^2 \sin \frac{1}{x} \right| = |x^2| \cdot \left| \sin \frac{1}{x} \right| \le |x^2| \cdot 1 = x^2$$

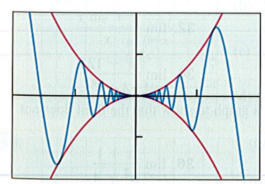
and

$$-x^2 \le x^2 \sin \frac{1}{x} \le x^2.$$

Because $\lim_{x \to 0} (-x^2) = \lim_{x \to 0} x^2 = 0$, the Sandwich Theorem gives

$$\lim_{x \to 0} \left( x^2 \sin \frac{1}{x} \right) = 0.$$

The graphs in Figure 2.8 support this result.



[−0.2, 0.2] by [−0.02, 0.02]

**Figure 2.8** The graphs of $y_1 = x^2$, $y_2 = x^2 \sin(1/x)$, and $y_3 = -x^2$. Notice that $y_3 \le y_2 \le y_1$. (Example 9)

## Quick Review 2.1 *(For help, go to Section 1.2.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–4, find $f(2)$.

**1.** $f(x) = 2x^3 - 5x^2 + 4$

**2.** $f(x) = \dfrac{4x^2 - 5}{x^3 + 4}$

**3.** $f(x) = \sin\left(\pi \dfrac{x}{2}\right)$

**4.** $f(x) = \begin{cases} 3x - 1, & x < 2 \\ \dfrac{1}{x^2 - 1}, & x \ge 2 \end{cases}$

In Exercises 5–8, write the inequality in the form $a < x < b$.

**5.** $|x| < 4$

**6.** $|x| < c^2$

**7.** $|x - 2| < 3$

**8.** $|x - c| < d^2$

In Exercises 9 and 10, write the fraction in reduced form.

**9.** $\dfrac{x^2 - 3x - 18}{x + 3}$

**10.** $\dfrac{2x^2 - x}{2x^2 + x - 1}$

## Section 2.1 Exercises

In Exercises 1–4, an object dropped from rest from the top of a tall building falls $y = 16t^2$ feet in the first $t$ seconds.

**1.** Find the average speed during the first 3 seconds of fall.

**2.** Find the average speed during the first 4 seconds of fall.

**3.** Find the speed of the object at $t = 3$ seconds and confirm your answer algebraically.

**4.** Find the speed of the object at $t = 4$ seconds and confirm your answer algebraically.

In Exercises 5 and 6, use $\lim_{x \to c} k = k$, $\lim_{x \to c} x = c$, and the properties of limits to find the limit.

**5.** $\lim_{x \to c} (2x^3 - 3x^2 + x - 1)$

**6.** $\lim_{x \to c} \dfrac{x^4 - x^3 + 1}{x^2 + 9}$

In Exercises 7–14, determine the limit by substitution. Support graphically.

**7.** $\lim_{x \to -1/2} 3x^2(2x - 1)$

**8.** $\lim_{x \to -4} (x + 3)^{1998}$

**9.** $\lim_{x \to 1} (x^3 + 3x^2 - 2x - 17)$

**10.** $\lim_{y \to 2} \dfrac{y^2 + 5y + 6}{y + 2}$

**11.** $\lim_{y \to -3} \dfrac{y^2 + 4y + 3}{y^2 - 3}$

**12.** $\lim_{x \to 1/2} \text{int } x$

**13.** $\lim_{x \to -2} (x - 6)^{2/3}$

**14.** $\lim_{x \to 2} \sqrt{x + 3}$

In Exercises 15–20, complete the following tables and state what you believe $\lim_{x \to 0} f(x)$ to be.

**(a)**

| $x$ | −0.1 | −0.01 | −0.001 | −0.0001 | … |
|---|---|---|---|---|---|
| $f(x)$ | ? | ? | ? | ? | |

**(b)**

| $x$ | 0.1 | 0.01 | 0.001 | 0.0001 | … |
|---|---|---|---|---|---|
| $f(x)$ | ? | ? | ? | ? | |

**15.** $f(x) = \dfrac{x^2 + 6x + 2}{x + 1}$

**16.** $f(x) = \dfrac{x^2 - x}{x}$

**17.** $f(x) = x \sin \dfrac{1}{x}$

**18.** $f(x) = \sin \dfrac{1}{x}$

**19.** $f(x) = \dfrac{10^x - 1}{x}$

**20.** $f(x) = x \sin (\ln |x|)$

In Exercises 21–24, explain why you cannot use substitution to determine the limit. Find the limit if it exists.

**21.** $\lim_{x \to -2} \sqrt{x - 2}$

**22.** $\lim_{x \to 0} \dfrac{1}{x^2}$

**23.** $\lim_{x \to 0} \dfrac{|x|}{x}$

**24.** $\lim_{x \to 0} \dfrac{(4 + x)^2 - 16}{x}$

In Exercises 25–34, determine the limit graphically. Confirm algebraically.

**25.** $\lim_{x \to 1} \dfrac{x - 1}{x^2 - 1}$

**26.** $\lim_{t \to 2} \dfrac{t^2 - 3t + 2}{t^2 - 4}$

**27.** $\lim_{x \to 0} \dfrac{5x^3 + 8x^2}{3x^4 - 16x^2}$

**28.** $\lim_{x \to 0} \dfrac{\frac{1}{2 + x} - \frac{1}{2}}{x}$

**29.** $\lim_{x \to 0} \dfrac{(2 + x)^3 - 8}{x}$

**30.** $\lim_{x \to 0} \dfrac{\sin 2x}{x}$

**31.** $\lim_{x \to 0} \dfrac{\sin x}{2x^2 - x}$

**32.** $\lim_{x \to 0} \dfrac{x + \sin x}{x}$

**33.** $\lim_{x \to 0} \dfrac{\sin^2 x}{x}$

**34.** $\lim_{x \to 5} \dfrac{x^3 - 125}{x - 5}$

In Exercises 35 and 36, use a graph to show that the limit does not exist.

**35.** $\lim_{x \to 1} \dfrac{x^2 - 4}{x - 1}$

**36.** $\lim_{x \to 2} \dfrac{x + 1}{x^2 - 4}$

In Exercises 37–42, determine the limit.

**37.** $\lim_{x \to 0^+} \text{int } x$

**38.** $\lim_{x \to 0^-} \text{int } x$

**39.** $\lim_{x \to 0.01} \text{int } x$

**40.** $\lim_{x \to 2^-} \text{int } x$

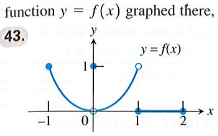
**41.** $\lim_{x \to 0^+} \dfrac{x}{|x|}$

**42.** $\lim_{x \to 0^-} \dfrac{x}{|x|}$



In Exercises 43 and 44, which of the statements are true about the function $y = f(x)$ graphed there, and which are false?

**43.**




(a) $\lim_{x \to -1^+} f(x) = 1$

(b) $\lim_{x \to 0^-} f(x) = 0$

(c) $\lim_{x \to 0^-} f(x) = 1$

(d) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

(e) $\lim_{x \to 0} f(x)$ exists

(f) $\lim_{x \to 0} f(x) = 0$

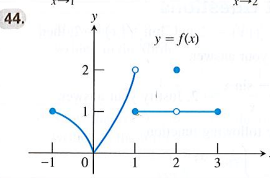
(g) $\lim_{x \to 0} f(x) = 1$

(h) $\lim_{x \to 1} f(x) = 1$

(i) $\lim_{x \to 1} f(x) = 0$

(j) $\lim_{x \to 2^-} f(x) = 2$

**44.**




(a) $\lim_{x \to -1^+} f(x) = 1$

(b) $\lim_{x \to 2} f(x)$ does not exist.

(c) $\lim_{x \to 2} f(x) = 2$

(d) $\lim_{x \to 1^-} f(x) = 2$

(e) $\lim_{x \to 1^+} f(x) = 1$

(f) $\lim_{x \to 1} f(x)$ does not exist.

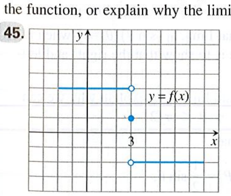
(g) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$

(h) $\lim_{x \to c} f(x)$ exists at every $c$ in $(-1, 1)$.

(i) $\lim_{x \to c} f(x)$ exists at every $c$ in $(1, 3)$.

In Exercises 45–50, use the graph to estimate the limits and value of the function, or explain why the limits do not exist.

**45.**



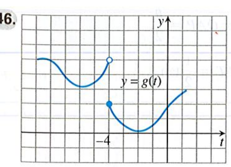

(a) $\lim_{x \to 3^-} f(x)$

(b) $\lim_{x \to 3^+} f(x)$

(c) $\lim_{x \to 3} f(x)$

(d) $f(3)$

**46.**



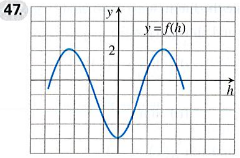

(a) $\lim_{t \to -4^-} g(t)$

(b) $\lim_{t \to -4^+} g(t)$

(c) $\lim_{t \to -4} g(t)$

(d) $g(-4)$

**47.**



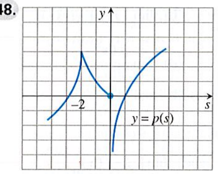

(a) $\lim_{h \to 0^-} f(h)$

(b) $\lim_{h \to 0^+} f(h)$

(c) $\lim_{h \to 0} f(h)$

(d) $f(0)$

**48.**



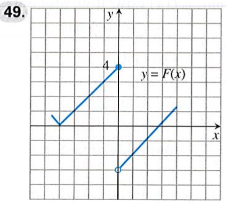

(a) $\lim_{s \to -2^-} p(s)$

(b) $\lim_{s \to -2^+} p(s)$

(c) $\lim_{s \to -2} p(s)$

(d) $p(-2)$

**49.**



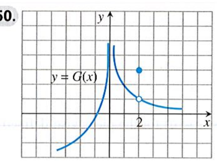

(a) $\lim_{x \to 0^-} F(x)$

(b) $\lim_{x \to 0^+} F(x)$

(c) $\lim_{x \to 0} F(x)$

(d) $F(0)$

**50.**




(a) $\lim_{x \to 2^-} G(x)$

(b) $\lim_{x \to 2^+} G(x)$

(c) $\lim_{x \to 2} G(x)$

(d) $G(2)$

In Exercises 51–54, match the function with the table.

**51.** $y_1 = \dfrac{x^2 + x - 2}{x - 1}$

**52.** $y_1 = \dfrac{x^2 - x - 2}{x - 1}$

**53.** $y_1 = \dfrac{x^2 - 2x + 1}{x - 1}$

**54.** $y_1 = \dfrac{x^2 + x - 2}{x + 1}$

| X | Y₁ |
|---|---|
| .7 | −.4765 |
| .8 | −.3111 |
| .9 | −.1526 |
| 1 | 0 |
| 1.1 | .14762 |
| 1.2 | .29091 |
| 1.3 | .43043 |
| X = .7 | |

(a)

| X | Y₁ |
|---|---|
| .7 | 7.3667 |
| .8 | 10.8 |
| .9 | 20.9 |
| 1 | ERROR |
| 1.1 | −18.9 |
| 1.2 | −8.8 |
| 1.3 | −5.367 |
| X = .7 | |

(b)

| X | Y₁ |
|---|---|
| .7 | 2.7 |
| .8 | 2.8 |
| .9 | 2.9 |
| 1 | ERROR |
| 1.1 | 3.1 |
| 1.2 | 3.2 |
| 1.3 | 3.3 |
| X = .7 | |

(c)

| X | Y₁ |
|---|---|
| .7 | −.3 |
| .8 | −.2 |
| .9 | −.1 |
| 1 | ERROR |
| 1.1 | .1 |
| 1.2 | .2 |
| 1.3 | .3 |
| X = .7 | |

(d)

In Exercises 55 and 56, determine the limit.

**55.** Assume that $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = 3$.

(a) $\lim_{x\to 4} (g(x) + 3)$ (b) $\lim_{x\to 4} x f(x)$

(c) $\lim_{x\to 4} g^2(x)$ (d) $\lim_{x\to 4} \dfrac{g(x)}{f(x) - 1}$

**56.** Assume that $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$.

(a) $\lim_{x\to b} (f(x) + g(x))$ (b) $\lim_{x\to b} (f(x) \cdot g(x))$

(c) $\lim_{x\to b} 4\, g(x)$ (d) $\lim_{x\to b} \dfrac{f(x)}{g(x)}$

In Exercises 57–60, complete parts (a), (b), and (c) for the piecewise-defined function.

(a) Draw the graph of $f$.

(b) Determine $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$.

(c) **Writing to Learn** Does $\lim_{x\to c} f(x)$ exist? If so, what is it? If not, explain.

**57.** $c = 2, f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$

**58.** $c = 2, f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ x/2, & x > 2 \end{cases}$

**59.** $c = 1, f(x) = \begin{cases} \dfrac{1}{x - 1}, & x < 1 \\ x^3 - 2x + 5, & x \ge 1 \end{cases}$

**60.** $c = -1, f(x) = \begin{cases} 1 - x^2, & x \ne -1 \\ 2, & x = -1 \end{cases}$

In Exercises 61–64, complete parts (a)–(d) for the piecewise-defined function.

(a) Draw the graph of $f$.

(b) At what points $c$ in the domain of $f$ does $\lim_{x\to c} f(x)$ exist?

(c) At what points $c$ does only the left-hand limit exist?

(d) At what points $c$ does only the right-hand limit exist?

**61.** $f(x) = \begin{cases} \sin x, & -2\pi \le x < 0 \\ \cos x, & 0 \le x \le 2\pi \end{cases}$

**62.** $f(x) = \begin{cases} \cos x, & -\pi \le x < 0 \\ \sec x, & 0 \le x \le \pi \end{cases}$

**63.** $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

**64.** $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1, \text{ or } x > 1 \end{cases}$

In Exercises 65–68, find the limit graphically. Use the Sandwich Theorem to confirm your answer.

**65.** $\lim_{x\to 0} x \sin x$ **66.** $\lim_{x\to 0} x^2 \sin x$

**67.** $\lim_{x\to 0} x^2 \sin \dfrac{1}{x^2}$ **68.** $\lim_{x\to 0} x^2 \cos \dfrac{1}{x^2}$

**69.** ***Free Fall*** A water balloon dropped from a window high above the ground falls $y = 4.9t^2$ m in $t$ sec. Find the balloon's

(a) average speed during the first 3 sec of fall.

(b) speed at the instant $t = 3$.

**70.** ***Free Fall on a Small Airless Planet*** A rock released from rest to fall on a small airless planet falls $y = gt^2$ m in $t$ sec, $g$ a constant. Suppose that the rock falls to the bottom of a crevasse 20 m below and reaches the bottom in 4 sec.

(a) Find the value of $g$.

(b) Find the average speed for the fall.

(c) With what speed did the rock hit the bottom?

## Standardized Test Questions

**71. True or False** If $\lim_{x\to c^-} f(x) = 2$ and $\lim_{x\to c^+} f(x) = 2$, then $\lim_{x\to c} f(x) = 2$. Justify your answer.

**72. True or False** $\lim_{x\to 0} \dfrac{x + \sin x}{x} = 2$. Justify your answer.

In Exercises 73–76, use the following function.

$$f(x) = \begin{cases} 2 - x, & x \le 1 \\ \dfrac{x}{2} + 1, & x > 1 \end{cases}$$

**73. Multiple Choice** What is the value of $\lim_{x\to 1^-} f(x)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

**74. Multiple Choice** What is the value of $\lim_{x\to 1^+} f(x)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

**75. Multiple Choice** What is the value of $\lim_{x\to 1} f(x)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

**76. Multiple Choice** What is the value of $f(1)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

**77. Group Activity** To prove that $\lim_{\theta\to 0} (\sin\theta)/\theta = 1$ when $\theta$ is measured in radians, the plan is to show that the right- and left-hand limits are both 1.

(a) To show that the right-hand limit is 1, explain why we can restrict our attention to $0 < \theta < \pi/2$.

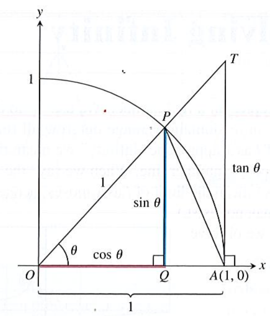
(b) Use the figure to show that

$$\text{area of } \Delta OAP = \frac{1}{2}\sin\theta,$$

$$\text{area of sector } OAP = \frac{\theta}{2},$$

$$\text{area of } \Delta OAT = \frac{1}{2}\tan\theta.$$






(c) Use part (b) and the figure to show that for $0 < \theta < \pi/2$,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

(d) Show that for $0 < \theta < \pi/2$ the inequality of part (c) can be written in the form

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

(e) Show that for $0 < \theta < \pi/2$ the inequality of part (d) can be written in the form

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

(f) Use the Sandwich Theorem to show that

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$

(g) Show that $(\sin\theta)/\theta$ is an even function.

(h) Use part (g) to show that

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1.$$

(i) Finally, show that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

## Extending the Ideas

**78.** ***Controlling Outputs*** Let $f(x) = \sqrt{3x - 2}$.

(a) Show that $\lim_{x\to 2} f(x) = 2 = f(2)$.

(b) Use a graph to estimate values for $a$ and $b$ so that $1.8 < f(x) < 2.2$ provided $a < x < b$.

(c) Use a graph to estimate values for $a$ and $b$ so that $1.99 < f(x) < 2.01$ provided $a < x < b$.

**79.** ***Controlling Outputs*** Let $f(x) = \sin x$.

(a) Find $f(\pi/6)$.

(b) Use a graph to estimate an interval $(a, b)$ about $x = \pi/6$ so that $0.3 < f(x) < 0.7$ provided $a < x < b$.

(c) Use a graph to estimate an interval $(a, b)$ about $x = \pi/6$ so that $0.49 < f(x) < 0.51$ provided $a < x < b$.

**80.** ***Limits and Geometry*** Let $P(a, a^2)$ be a point on the parabola $y = x^2$, $a > 0$. Let $O$ be the origin and $(0, b)$ the $y$-intercept of the perpendicular bisector of line segment $OP$. Find $\lim_{P\to O} b$.